A k-EXTREME POINT IS THE LIMIT OF k-EXPOSED POINTS

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ABSTRACT

It is proved that the relative boundary of a k-dimensional intersection of a hyperplane and a compact convex body is contained in the closure of the union of all intersections of dimension lower than p that the same convex body makes with different hyperplanes.

We prove in this note a theorem that generalizes a theorem of Straszewicz [3]. Let C be a compact convex body in R^n.

DEFINITION 1. A point $p \in C$ is called k-extreme if it is not the centroid of some non-degenerate $(k+1)$-simplex in C.

In the terminology of Bourbaki [2, §1, exerc. 2] one would say that a k-extreme point is of order at most k.

DEFINITION 2. A point $p \in C$ is called k-exposed if it is contained in a closed half-space K such that $K \cap C$ is at most k-dimensional.

We collect some immediate consequences of the definitions.

COROLLARY. *A k-exposed point is k-extreme. A k-exposed (extreme) point is h-exposed (extreme) for all $h > k$. If ϕ is a supporting hyperplane of C and $p \in \phi \cap C$ is k-extreme with respect to $\phi \cap C$, then p is k-extreme with respect to C.*

We will call 0-exposed (extreme) points exposed (extreme) to conform with earlier usage. The following theorem coincides for $k = 0$ with the theorem of Straszewicz.

THEOREM. *For $k \geqq 0$, the closure of the set of k-exposed points contains the set of k-extreme points.*

For $k \geqq n-1$, the theorem is true trivially. For $k = n-1$ the set of k-exposed points and the set of k-extreme points both coincide with bd C and for $k \geqq n$ with C.

Received September 30, 1963

This theorem is applied in [1] to the problem of determining which subsets of a general finite-dimensional Banach space have unique farthest points.

We proceed to prove the theorem by induction on k. The proof for $k=0$ may be found in Straszewicz [3] and in Bourbaki [2, §4, exerc. 15c], and we will use this "ordinary Straszewicz theorem" both as a starting point for the induction and in the proof itself.

Suppose that the theorem has been proved for $k < h$. Let p be a point in the complement of the closure of the set of h-exposed points. In order to obtain a contradiction we also add the hypothesis that p is h-extreme.

Let A be an open neighborhood of p such that no $q \in A$ is h-exposed. By the induction hypothesis, p is the centroid of some non-degenerate h-simplex S. Choose a linear manifold M that passes through p and that is supplementary to the one generated by S. Since by hypothesis the point p is h-extreme, it must be extreme relative to $M \cap C$ and hence, by the ordinary Straszewicz theorem, some point $q \in A \cap M \cap C$ is exposed relative to $M \cap C$.

We will now show that the face of q is at most $(h-1)$-dimensional, thereby arriving at the contradiction, since by hypothesis q is not $(h-1)$-extreme.

Let N be a hyperplane of M that separates q from $M \cap C$. As an affine manifold of R^n, N has codimension $h+1$ and $N \cap C = \{q\}$. By the Hahn-Banach theorem, there is a hyperplane ϕ containing N that supports C at q. In this hyperplane N has codimension h and separates q from the rest of $\phi \cap C$. But q is not h-exposed, hence $\phi \cap C$ has at least dimension $h+1$.

Let F be the affine space generated by $\phi \cap C$. Then $N \cap F$ has codimension at most h in F, and $(N \cap F) \cap C = \{q\}$ so that there exists a hyperplane ψ of F supporting $\phi \cap C = F \cap C$ at q and containing $N \cap F$. In ψ, $N \cap F$ has codimension at most $h-1$. But ψ also contains the face of q (in C), which therefore can have at most dimension $h-1$. The proof is now complete.

References

1. Asplund, E., Sets with unique farthest points (to appear).
2. Bourbaki, N., 1956, Espaces vectoriels topologiques, chapitre IV. Paris.
3. Straszewicz, S., 1935, Über exponierte Punkte abgeschlossener Punktmengen, *Fund. Math.*, **24**, 139–143.

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